Fuzzy Inference and Defuzzification

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Lecture Outline
- Crisp Rules Revision
- Fuzzy Sets revision
- Fuzzy Inference
- Fuzzy Rules
- Fuzzy Composition
- Defuzzification

Crisp Rules
- Consist of antecedents and consequents
- Each part of an antecedent is a logical expression
  - e.g. A > 0.5, light is on
- Consequent will be asserted if antecedent is true
  - IF (Presentation is Dull) AND (Voice is Monotone)
  - THEN Lecture is boring

Crisp Rules
- Only one rule at a time allowed to fire
- A rule will either fire or not fire
- Have problems with uncertainty
- Have problems with representing concepts like small, large, thin, wide
- Sequential firing of rules also a problem
  - order of firing

Fuzzy Sets
- Supersets of crisp sets
- Items can belong to varying degrees
  - degrees of membership
  - [0,1]
- Fuzzy sets defined two ways
  - membership functions
    - MF
  - sets of ordered pairs

Fuzzy Sets
- Membership functions (MF)
- Mathematical functions
- Return the degree of membership in a fuzzy set
- Many different types in existence
  - Gaussian
  - Triangular
Fuzzy Sets

• Can also be described as sets of ordered pairs
• Pair Crisp, Fuzzy values
  – \( A = \{(0,1.0), (1,1.0), (2,0.75), (3,0.5), (4,0.25), (5,0.0), (6,0.0), (7,0.0), (8,0.0), (9,0.0), (10,0.0)\} \)
• With enough pairs, can approximate any MF

Fuzzy Sets

• Fuzzification
• Process of finding the degree of membership of a value in a fuzzy set
• Can be done by
  – MF
  – Interpolating set of pairs

Fuzzy Rules

• Also have antecedents and consequents
• Both deal with partial truths
• Antecedents match fuzzy sets
• Consequents assign fuzzy sets
• Fuzzy rules can have weightings
  – \([0,1]\)
  – importance of rule
  – commonly set to 1

Fuzzy Rules

• Restaurant tipping example
• Antecedent variables are
  – quality of service
  – quality of food
• Consequent variables are
  – Tip

Fuzzy Rules

• Service can be
  – Poor
  – good
  – excellent
• Universe of discourse is 0-10

Fuzzy Rules

• Food can be
  – rancid
  – good
  – delicious
• Universe of discourse is 0-10
Fuzzy Rules

- Tip can be
  - cheap
  - average
  - generous
- Universe of discourse is 0-25%

Fuzzy Inference

- Infers fuzzy conclusions from fuzzy facts
- Matches facts against fuzzy antecedents
- Assigns fuzzy sets to outputs
- Three step process
  - fuzzify the inputs (fuzzification)
  - apply fuzzy logical operators across antecedents
  - apply implication method

Fuzzy Inference

- Rules for the tipping system
  - IF service is poor or food is rancid
  - THEN tip is cheap
  - IF service is good
  - THEN tip is average
  - IF service is excellent or food is delicious
  - THEN tip is generous

Fuzzy Inference

- Implication is really two different processes
  - inference
  - composition
- Inference is the matching of facts to antecedents
- Results in the truth value of each rule
  - degree of support
  - Alpha

Fuzzy Inference

- Assigns fuzzy sets to each output variable
- Fuzzy sets assigned to different degrees
- Determined by degree of support for rule
- Methods for assigning (inferring) sets
  - min
  - Product

Fuzzy Inference

- Min inference
- Cut output MF at degree of support

\[ \mu(v)' = \min(z, \mu(v)) \]

Where:
- \( \mu \) the output MF
- \( \mu' \) is the inferred MF
- \( v \) is the value being fuzzified
- \( z \) is the degree of support
Fuzzy Inference

- Product inferencing
- Multiply output MF by degree of support
  \[ \mu(v)' = z\mu(v) \]

Tipping Example

- Assume
  - service is poor
    - score of 2
  - food is delicious
    - score of 8
- How do we perform fuzzy inference with these values?

Tipping Example

- Firstly, fuzzify the input values
- Service fuzzifies to
  - Poor 0.8
  - Good 0.2
  - Excellent 0.0
- Food fuzzifies to
  - Rancid 0.0
  - Good 0.4
  - Delicious 0.6

Tipping Example

- Now, calculate the degree of support for each rule
- Rule 1:
  - IF service is poor or food is rancid
    - poor = 0.8
    - rancid = 0.0
    - \( \max(0.8, 0.0) = 0.8 \)
    - Degree of support = 0.8

Tipping Example

- Rule 2
  - IF service is good
    - good = 0.2
    - \( \max(0.2) = 0.2 \)
    - Degree of support = 0.2

Tipping Example

- Rule 3
  - IF service is excellent or food is delicious
    - excellent = 0.0
    - delicious = 0.6
    - \( \max(0.0, 0.6) = 0.6 \)
    - Degree of support = 0.6
Tipping Example

- Apply implication method
- Builds an inferred fuzzy set
- Find the min value for each output MF
- Cut output MF at this value

Min Inference

- Cut at 0.8

Min Inference

- Corresponding fuzzy set

\[ MF = \{(0,0),(1,0.2),(2,0.4),(3,0.6),(4,0.8),(5,0.8), (6,0.8),(7,0.6),(8,0.4),(9,0.2),(10,0),(25,0)\} \]

Min Inference

- Degree of support of 0.4
Min Inference

- Corresponding set
  - MF = {(0,0),(1,0.2),(2,0.4),(3,0.4),(4,0.4),(5,0.4),
           (6,0.4),(7,0.4),(8,0.4),(9,0.2),(10,0), (25,0)}

Fuzzy Inference

- How are things different if we use product inferencing?

Product Inference

- Corresponding set
  - MF = {(0,0),(1,0.16),(2,0.32),(3,0.48),(4,0.64),(5,0.8),
           (6,0.64),(7,0.48),(8,0.16),(9,0.16),(10,0), (25,0)}

Product Inference

- Degree of support of 0.4

Product Inference

- Corresponding set
  - MF = {(0,0),(1,0.08),(2,0.16),(3,0.24),(4,0.32),(5,0.4),
           (6,0.32),(7,0.24),(8,0.16),(9,0.08),(10,0), (25,0)}
Fuzzy Composition

- Aggregates the inferred MF into one
- Two methods of doing this
  - Max
  - Sum

Max Composition

- MAX composition
  - Take the max of each column

<table>
<thead>
<tr>
<th>v</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
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</thead>
<tbody>
<tr>
<td>$\mu(y)$</td>
<td>0.16</td>
<td>0.32</td>
<td>0.48</td>
<td>0.64</td>
<td>0.80</td>
<td>0.64</td>
<td>0.48</td>
<td>0.32</td>
<td>0.16</td>
<td>0</td>
<td></td>
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<table>
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<tr>
<th>v</th>
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<th>14</th>
<th>15</th>
<th>16</th>
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<th>18</th>
<th>19</th>
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<th>22</th>
<th>23</th>
<th>24</th>
<th>25</th>
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<tbody>
<tr>
<td>$\mu(y)$</td>
<td>0.48</td>
<td>0.36</td>
<td>0.24</td>
<td>0.14</td>
<td>0.06</td>
<td>0.02</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Max Composition

- MAX takes the max fuzzy value for each value of $v$
  - equivalent to taking the fuzzy values for the highest activated rule for each output fuzzy set
- SUM sums all fuzzy values for each value of $v$
  - can lead to truth values > 1
  - may need to be normalised to [0, 1]
    - implications for defuzzification

Fuzzy Composition

- Assume
  - 3 MF attached to the output
    - A, B and C
  - Each MF has been asserted by two different rules
    - 6 rules activated (degrees of support) > 0
  - Degrees of support
    - 0.8, 0.4, 0.6, 0.5, 0.7, 0.3
  - Prod inference used

Max Composition

- For Set A
  - $\mu(y)$
    - 0.16 0.32 0.48 0.64 0.80 0.64 0.48 0.32 0.16 0
  - $\mu(v)$
    - 0.12 0.24 0.36 0.48 0.60 0.48 0.36 0.24 0.12 0

- For Set B
  - $\mu(y)$
    - 0.16 0.32 0.48 0.64 0.80 0.64 0.48 0.32 0.16 0
  - $\mu(v)$
    - 0.12 0.24 0.36 0.48 0.60 0.48 0.36 0.24 0.12 0

- For Set C
  - $\mu(y)$
    - 0.16 0.32 0.48 0.64 0.80 0.64 0.48 0.32 0.16 0
  - $\mu(v)$
    - 0.12 0.24 0.36 0.48 0.60 0.48 0.36 0.24 0.12 0
Sum Composition

- Sum composition
  - sum each column

<table>
<thead>
<tr>
<th>$\mu(t)$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
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<tbody>
<tr>
<td>$\sigma$</td>
<td>0.16</td>
<td>0.32</td>
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<td>0.64</td>
<td>0.8</td>
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<td>0.68</td>
<td>0.68</td>
<td>1.02</td>
<td>1.36</td>
<td>1.7</td>
<td></td>
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<tr>
<td>$\mu(t)$</td>
<td>1.36</td>
<td>1.02</td>
<td>0.68</td>
<td>0.56</td>
<td>0.68</td>
<td>1.02</td>
<td>1.36</td>
<td>1.7</td>
<td>1.36</td>
<td>1.02</td>
<td>0.68</td>
<td>0.34</td>
<td>0</td>
</tr>
</tbody>
</table>

Defuzzification

- Converts inferred MF into crisp numbers
- Many different types in existence
- Two common ones
  - Centre of Gravity
  - Mean of Maxima

COG Defuzzification

- Centre of Gravity
  - $y = \frac{\sum_{i}^{K} \mu(v_i)v_i}{\sum_{i}^{K} \mu(v_i)}$
- Where:
  - $y$ is the crisp value
  - $K$ is the number of items in the fuzzy set

COG Defuzzification

- Applying this to the first composite set

\[
\begin{align*}
\mu(t) & = 0.48, 0.26, 0.14, 0.26, 0.42, 0.36, 0.7, 0.36, 0.42, 0.28, 0.14, 0 \\
\sigma & = 6.24, 5.04, 3.6, 3.34, 3.78, 7.36, 7.36, 10.64, 14, 11.76, 0.26, 6.44, 3.36, 0
\end{align*}
\]

\[
\sum_{i}^{K} \mu(v_i)v_i = 121.68
\]

\[
\sum_{i}^{K} \mu(v_i) = 10.1
\]

\[
\frac{121.68}{10.1} = 12.05
\]
COG Defuzzification

- Mean of Maxima
  - MoM
- Finds the mean of the crisp values that correspond to the maximum fuzzy values
- If there is one maximum fuzzy value, the corresponding crisp value will be taken from the fuzzy set

MoM Defuzzification

- Applying this to the first composite set
- Maximum fuzzy value is 0.8
- Corresponding crisp value is 4
- This is the value returned by MoM

MoM Defuzzification

- What about sets with > 1 maximum?
- Apply this to the third composite set

<table>
<thead>
<tr>
<th>( \nu )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
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</thead>
<tbody>
<tr>
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<td>0.2</td>
<td>0.4</td>
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<td>0.8</td>
<td>0.8</td>
<td>0.7</td>
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<td>0.6</td>
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</tr>
<tr>
<td>( \nu )</td>
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</tr>
<tr>
<td>( \mu(\nu) )</td>
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<td>0.4</td>
<td>0.2</td>
<td>0.4</td>
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<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.6</td>
<td>0.4</td>
<td>0.2</td>
</tr>
</tbody>
</table>

MoM Defuzzification

- Maximum fuzzy value if 0.8
- Corresponding crisp values are
  - 4, 5 and 6
  
  \[ y = \frac{4 + 5 + 6}{3} = 5 \]
Summary

- Fuzzy rules match fuzzy antecedents to fuzzy consequents
- Degree to which antecedents are true determine the degree of support
- Fuzzy logic functions are used to determine this

Summary

- Fuzzy inference involves calculating an output fuzzy set
- Different inference process produces different inferred MF
- Two inferences processes are
  - max-min
  - Max-prod

Summary

- Two common composition methods
  - MAX
  - SUM
- Inference methods described by combining inference & composition methods
  - max-min (or min-max)
  - max-prod
- Defuzzification converts a composed MF to a single crisp value

Summary

- Different defuzzification methods produce different crisp values
  - sometimes wildly different
- Two different defuzzification methods
  - Centre of Gravity
    - CoG
  - Mean of Maxima
    - MoM